Duffing equation with two periodic forcings: The phase effect

Junzhong Yang,[1] Zhilin Qu,[1,2] and Gang Hu[1,2]
[1]*Department of Physics, Beijing Normal University, Beijing 100875, China*
[2]*Center of Theoretical Physics, Chinese Center of Advanced Science and Technology (World Laboratory), P.O. Box 8730, Beijing, China*
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A weak additional sinusoidal perturbation is applied to the periodically forced nonlinear oscillator to suppress chaos. Numerical simulations show that the phase difference between the two sinusoidal forces plays a very important role in controlling chaos. When the frequencies of these forces deviate from the resonance condition slightly, a different type of intermittency, alternation from regular motion to chaotic motion (called breather here), is observed. If the phase difference follows a Wiener process, conventional intermittency is observed.

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I. INTRODUCTION

The presence of chaos both in nature and in man-made devices is very common and has been extensively demonstrated in recent decades. Quite frequently chaos is a beneficial feature as in some chemical, heat, and transport problems [1]. However, in many other situations chaos is an undesirable phenomenon leading to irregular performance and possible catastrophic failures. The problem of controlling chaos (converting the chaotic motion to regular motion) has received considerable attention in recent years [2–20]. The controlling methods recently developed can be roughly classified into two categories: feedback and nonfeedback methods. Feedback methods [3–6] suppress chaos by stabilizing orbits already existing in the systems. Nonfeedback methods [7,8] apply small driving forces, or small modulations, directly to system parameters to suppress chaos; they modify the underlying dynamics and make stable solutions appear.

A typical nonfeedback control method can be generally modeled as

$$\dot{\vec{x}}=\vec{F}(\vec{x},B\ \cos(\omega t),\alpha\ \cos(\Omega t+\varphi)), \tag{1}$$

where the first periodic force drives the system to the chaotic state while the second one is a weak periodic force sensitively modifying the system dynamics. This kind of control has been investigated by many authors analytically [7], numerically [8,11], and experimentally [12,13]. However, to our knowledge, most of the previous studies of controlling chaos in such nonautonomous systems simply set phase $\varphi=0$; the role played by the phase difference φ has not been carefully studied to our knowledge. In this paper we will numerically investigate the effect of phase on suppressing chaos in detail along the line suggested in Ref. [15].

This paper is organized as follows. In Sec. II, we describe the results of numerical stimulation about the effect of phase on controlling chaos by applying sinusoidal perturbation to the Duffing equation. In Sec. III we describe the phenomenon of a different kind of intermittency, called breather in our paper, induced by frequency detuning. In Sec. IV, we discuss briefly the phase effect in multifrequency systems. All the results in this presentation are based on numerical simulations. Some heuristic explanations are provided for the breather phenomenon.

II. PHASE EFFECT ON SUPPRESSING CHAOS

In this section, we consider the problem of controlling chaos of the Duffing oscillator by a weak additional periodic perturbation,

$$\begin{aligned}\dot{x}&=y\\ \dot{y}&=-\delta y-x^3+B\ \cos\omega t+\alpha B\ \cos(\Omega t+\varphi).\end{aligned} \tag{2}$$

With $\alpha=0$ this equation is a usual Duffing oscillator (so-called soft spring oscillator), extensively investigated in nonlinear science such as chaos, plasma oscillations, and engineering problems [21]. δ is a dissipation parameter and $B\ \cos\omega t$ is the first forcing driving the system to the chaotic state. Apart from a single force, the effect of a number of competing external forcing frequencies on the region of chaos in the quasiperiodically forced Duffing oscillator has been investigated [22,23] recently. In this paper, we study the effect of suppressing chaos by the second weak force. Here, α is the amplitude of the perturbation and is assumed to be small, e.g., $\alpha\ll 1$. For certain combinations of parameters chaotic solutions can be obtained. Figure 1 shows the bifurcation of the Duffing equation [i.e., Eq. (2) with $\alpha=0$] versus B ($\delta=0.3$, $\omega=1$; these two parameters are fixed throughout this paper). We have adopted a fifth-order Runge-Kutta method to integrate Eq. (2) in the computer simulation. Data in Fig. 1 and throughout the presentation are taken on the following surface of section: $x=0$ and $y<0$; i.e., the surface of section is located on the negative y axis. Since there are multiple attractors coexisting for Eq. (2) in the parameter region investigated, we integrate Eq. (2) by employing the conventional technique that the terminal point of the orbit integrated for the former parameter is prepared as the starting point of the simulation for the sequential parameter to keep uniqueness of the results.

In Fig. 2 we show the bifurcation with respect to α for $B=8.85$, $\Omega=3$, $\varphi=0$. The results show that α plays a role as an additional relevant parameter for bifurcation and chaos.

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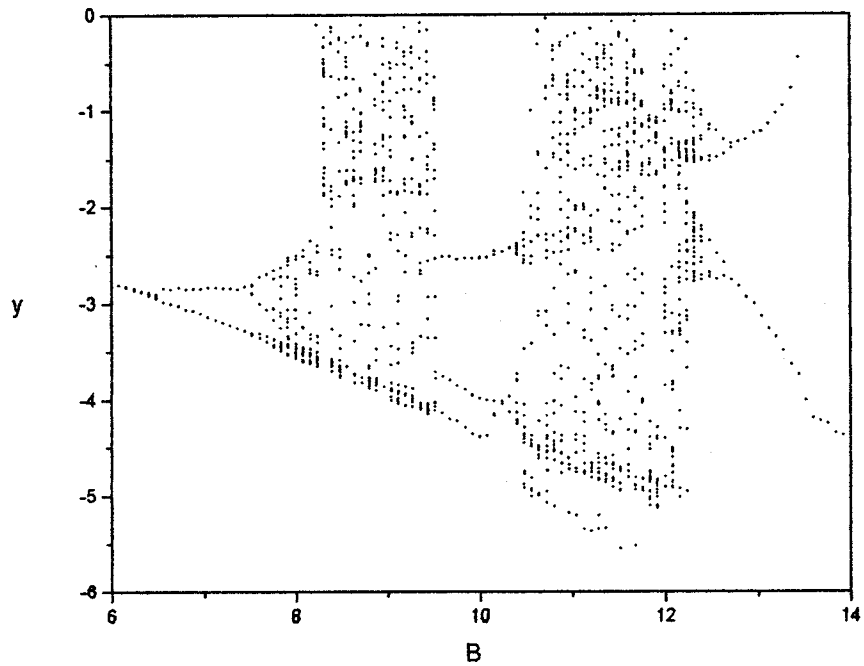

FIG. 1. Bifurcation diagrams of Eq. (1) with asymptotic y plotted against B. $\gamma=0.3$, $\omega=1$ (these parameters are taken in all the following figures), $\alpha=0$. The data are taken on the surface of section located on the negative y axis.

This point has been investigated in some previous papers, such as Refs. [7] and [8]. An interesting point is that in the small α region the external control forcing prefers to suppress chaos via inverse period-doubling bifurcation. However, in order to reach the inverse period-doubling bifurcation threshold, one has to vary the amplitude of the second forcing to a large extent, i.e., up to $\alpha\geqslant0.25$ in Fig. 2. This observation is rather disappointing in the sense of controlling chaos, since one has to apply a large external forcing to remove chaos. Without the perturbation $\alpha B\cos(\Omega t+\varphi)$ one can also achieve the same purpose of suppressing chaos in the original Duffing equation by changing the parameter, say B, in the same extent. By controlling, one expects that a small external control perturbation (in comparison with the original driving) should be able to bring the system out of the chaotic region; that is the main focus of this presentation.

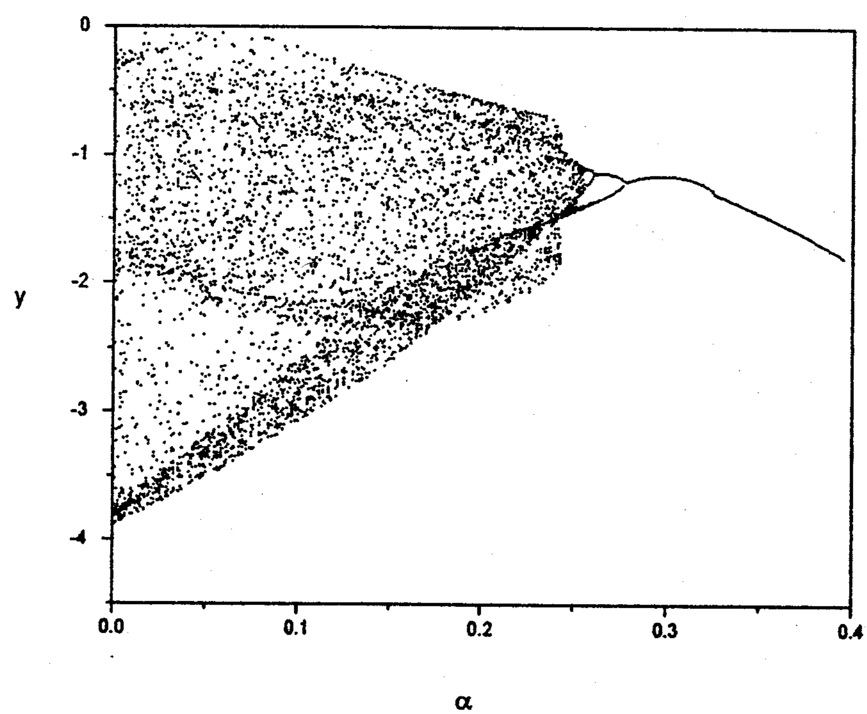

FIG. 2. Asymptotic y plotted against the perturbation intensity α. $B=8.85$, $\varphi=0$, $\Omega=3$.

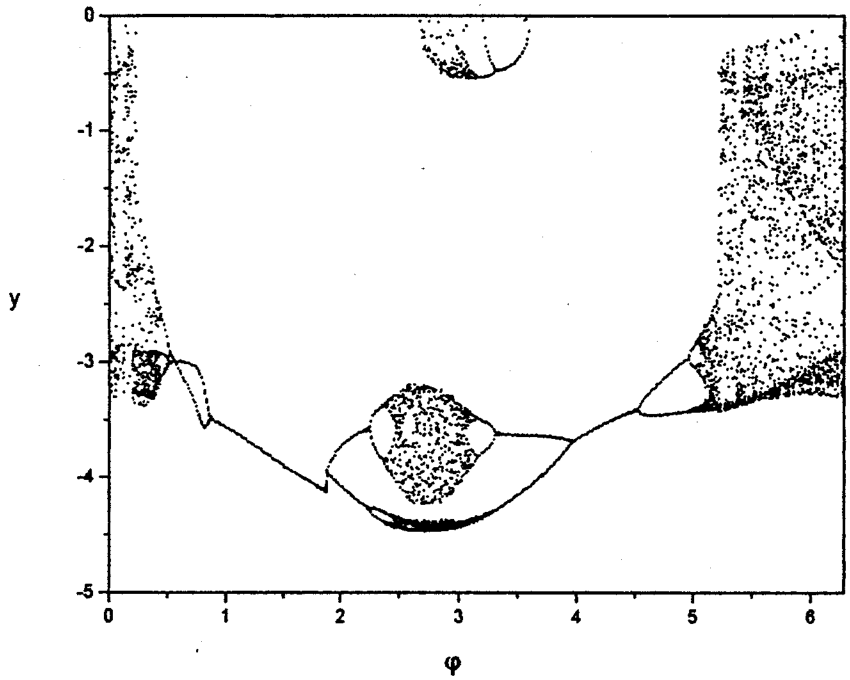

FIG. 3. Asymptotic y plotted against the phase φ for $B=8.85$, $\alpha=0.075$, $\Omega=3$.

Now, let us examine how the phase difference of the two forcings influences the bifurcation of the system. In Fig. 3, we show the bifurcation with respect to φ. The values of the parameters B and Ω are the same as those in Fig. 2. It is surprising that with the perturbation amplitude fixed at a rather small value the phase φ plays a very important role in suppressing chaos in each case. We find a wide range of phase producing regular motion, which connects chaos by period doubling or inverse period doubling. This observation indicates that the phase φ is a sensitive parameter for the system bifurcation; then one can suppress chaos by adding a weak external forcing with a proper phase to the system. The method of controlling chaos by adjusting the phase difference can be termed as phase control.

The above phase control concept has been drawn for certain isolated parameter combinations. To confirm this idea, we test the phase effect in wider parameter regions. First we fix $\alpha=0.06$ and $\Omega=3$ and consider the phase diagram in the

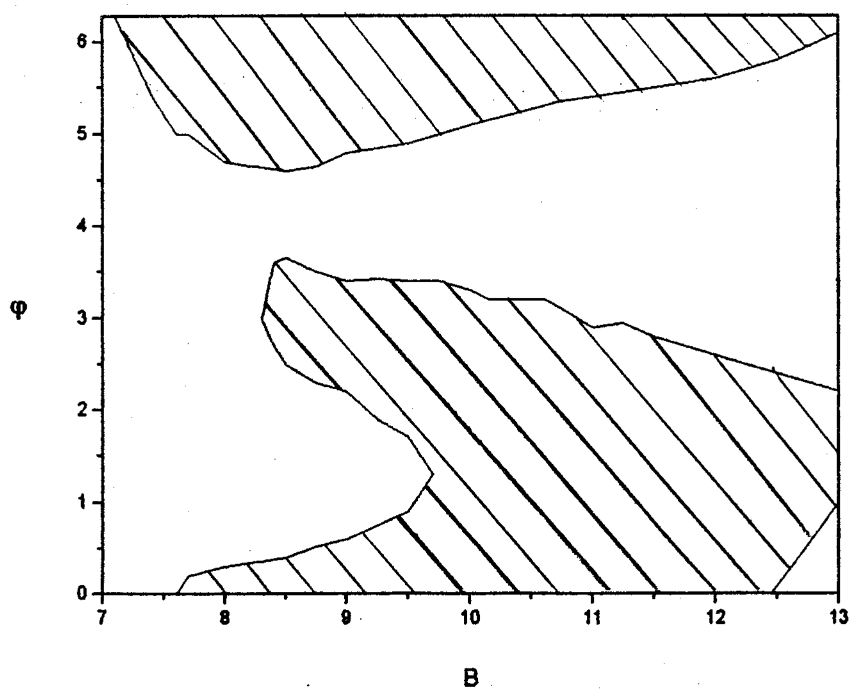

FIG. 4. Regular motion region (blank) and chaotic motion (including periodic windows) region (shaded) in the B-φ plane with $\alpha=0.06$; the boundary is defined by the bifurcation lines from period 4 to period 8.

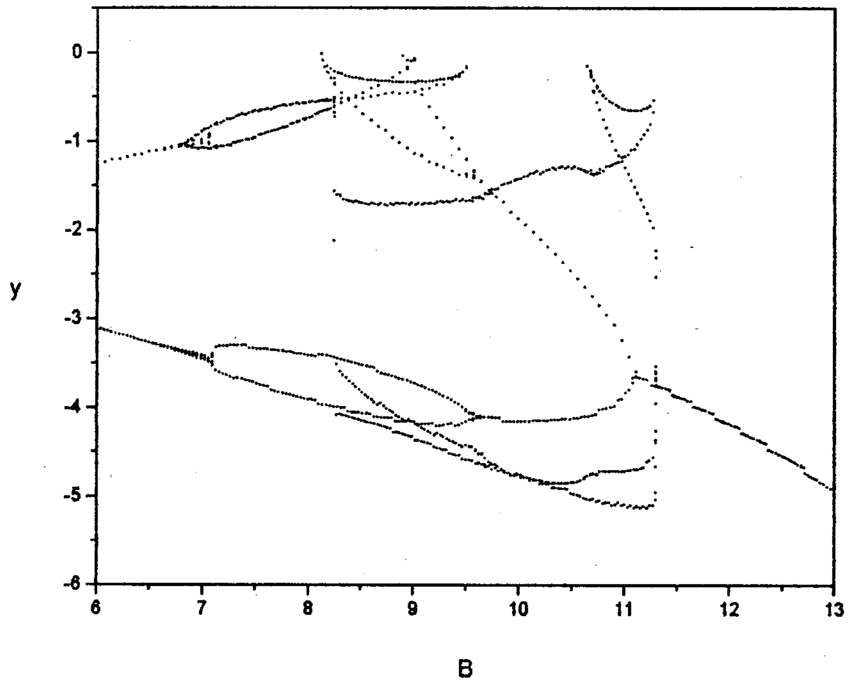

FIG. 5. The asymptotic y plotted against B. $\alpha=0.075$, $\Omega=3$, and $\varphi=3.53$. All attractors are presented. Two attractors coexist in $6<B<13$, three coexist in $8.3<B<11.2$. In the whole region from $B=6$ to $B=13$, chaos for all attractors is completely ruled out by applying a weak forcing.

B-φ plane. The numerical result is shown in Fig. 4, where we classify only regular motion (period 1,2,4, blank region) and chaotic motion (including period 8,16,... and other periodic windows, shaded region). In Fig. 4, the regular region, which leaves the chaotic region by inverse period bifurcation, is a connected region. It is extremely interesting that the distribution of regular and chaotic regions strongly depends on the phase difference. For certain φ (e.g., $3.5<\varphi<4.5$ in the figure), the chaos region completely vanishes in the entire B range where chaos exists without the second forcing (chaos is entirely wiped out rather than slightly shifted aside in this phase region). It is emphasized that in Fig. 4 the forcing

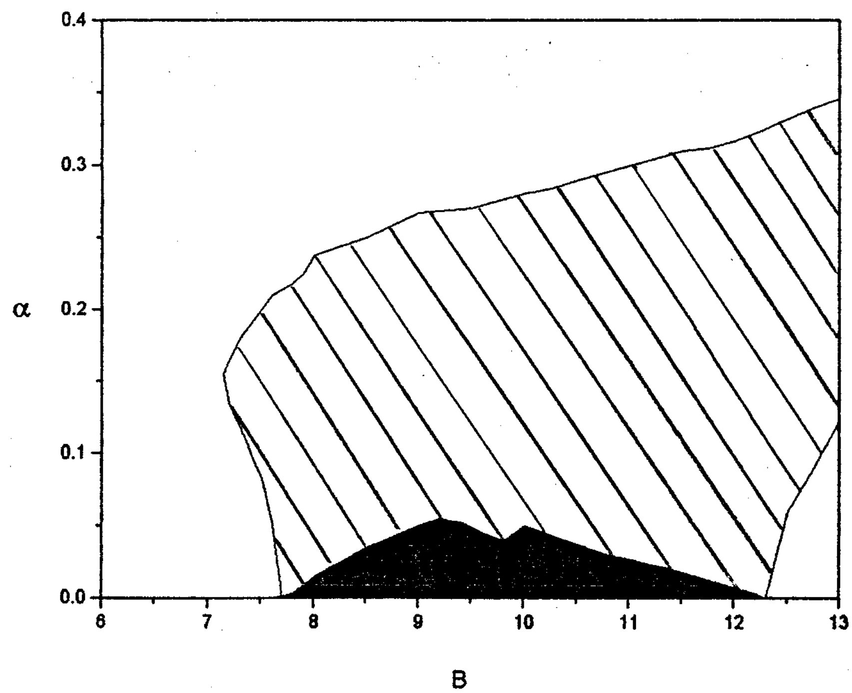

FIG. 6. At $\varphi=0$, blank region corresponds to regular states, shaded and black regions indicate chaotic states (including periodic windows). The black region is the uncontrollable region even when phase difference φ is taken into account. The black region considerably shrinks from the shaded region, which demonstrates the efficiency of phase control.

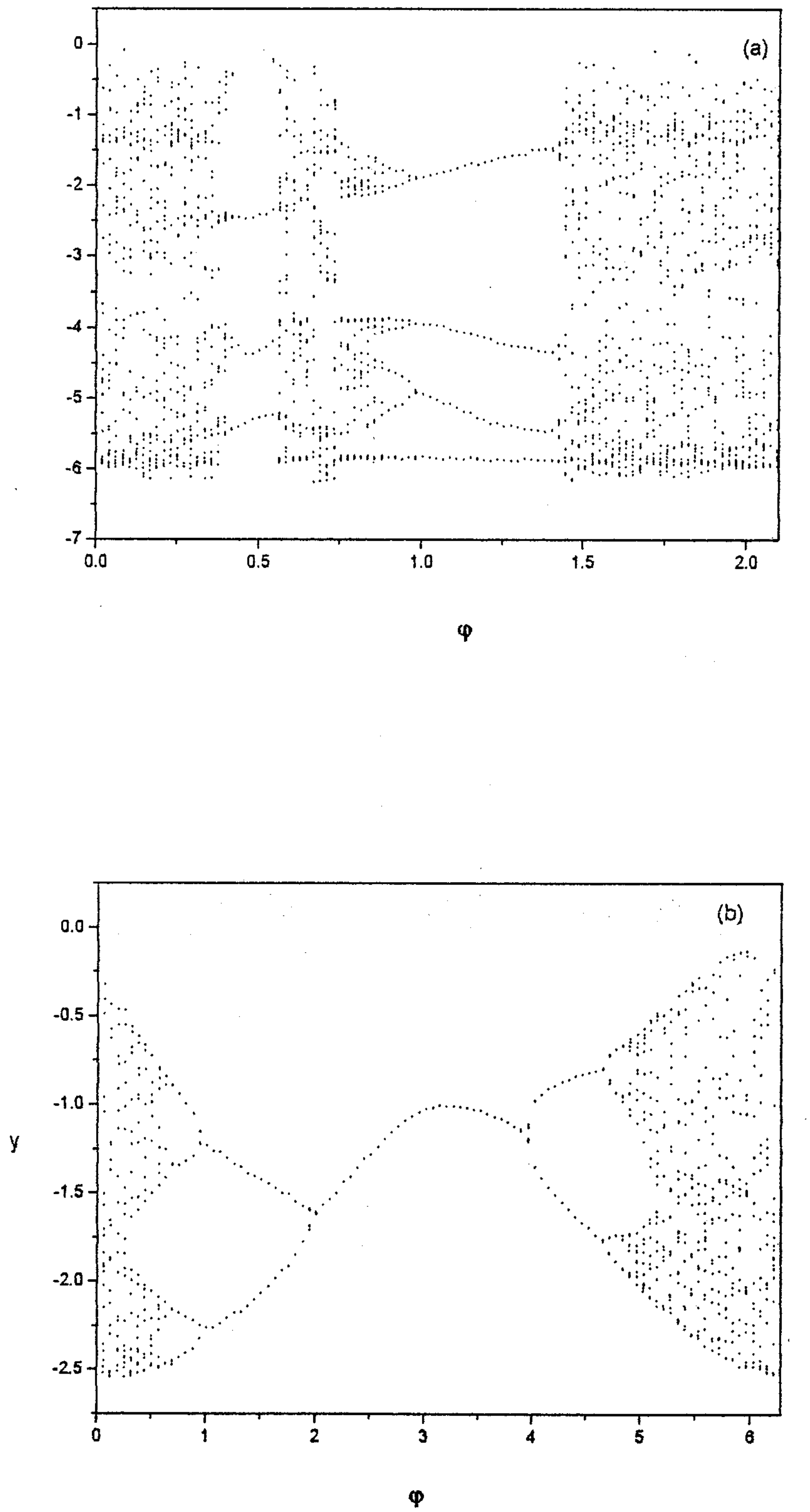

FIG. 7. Asymptotic y plotted against phase φ for different frequencies of the perturbation. (a) $\Omega=1/3$, $B=12.1$, and $\alpha=0.065$; (b) $\Omega=4$, $B=7.15$, and $\alpha=0.1$; (c) $\Omega=7$, $B=10.6$, and $\alpha=0.09$. In each case, regular motion can be found in a large interval of φ indicating that phase control is effective for different frequencies of the second forcing.

amplitude is rather small, much smaller than the threshold in Fig. 2 ($\alpha\approx0.25$) where $\varphi=0$ is taken. In Fig. 5, we take α and Ω the same as in Fig. 4 and fix $\varphi=3.53$ and plot the bifurcation diagram with respect to B. Note that the way to produce Fig. 5 is different from that for all the above figures. Here for each B we try different initial conditions and explore all attractors. We can clearly see that there are two attractors coexisting in $6<B<13$ and three in $8.3<B<11.2$. It is remarkable that only regular motions appear and the whole chaotic region is eliminated. This figure is considerably distinguished from Fig. 1 where the same B range is detected. A further investigation shows that for a narrow

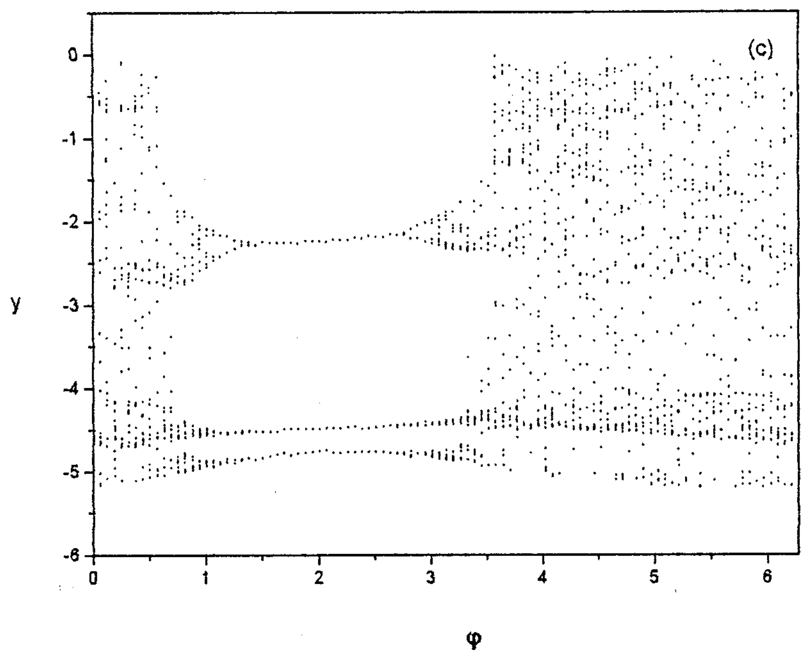

FIG. 7 (*Continued*).

range of phase all of the three attractors together can be controlled, while in a relatively wide φ range we can rule out chaos from two attractors.

In order to further demonstrate the effect of the phase difference on controlling chaos, we have studied the phase diagram in (α,B,φ) three-dimensional parameter space, but plot only the projection of the results in the B-α plane in Fig. 6. Figure 6 was produced in the following way. First we fix $\varphi=0$ and specify the bifurcation of the system in the B-α plane. The shaded and the black regions are chaotic regions (including the periodic windows) while the blank region represents the regular state, in the sense of Fig. 4, which leaves the shaded region by inverse period doubling. Then we change φ to obtain a minimum α with respect to φ at which the system can leave chaos by inverse period doubling for a given B. We plot the minimum α for each B by the boundary of the black region. Thus the blank and shaded regions together are the projection of all regular region (connected topologically) in B-α-φ space to the B-α plane. A remarkable feature is that the black region is considerably contracted from the shaded region and thus the threshold α of taming chaos can be very much reduced when the phase difference is taken into account.

In the above discussion, we consider $\Omega=3\omega$ only. Numerical simulations show that the features mentioned above are kept in the general case of $\Omega=\omega q/p$ with p and q being some incompensable integers. In Fig. 7 we show the bifurcation diagram of $\Omega=1/3$ (a), 4 (b), and 7 (c). In Fig. 7(a), we need to investigate only the range of $\varphi\in(0,2\pi/3)$, due to the symmetry property. From Fig. 7, it is obvious that in each case there is a wide range of φ in which chaotic motion can be converted to regular motion. Thus, phase control at weak second forcing is rather effective and can be applied generally.

To get a general idea about the influence of the frequency of the second forcing, we investigate the behavior of the leading Lyapunov exponent versus perturbative frequency. In Fig. 8(a), we show the Lyapunov exponent versus Ω for $\varphi=0$. Significant reduction of the value of λ is observed as Ω is switched to some resonant frequencies or in their neighbor. This feature has been mentioned by several authors [7,8]. However, the influence of the phase difference on λ for different frequencies has never been investigated to our knowledge. For example, as we take $\varphi=2\pi/3$, the λ-Ω curve in Fig. 8(b) becomes rather different from Fig. 8(a). In Fig. 8(c), we plot the $\lambda_{\min}$-Ω curve, where $\lambda_{\min}$ is the leading Lyapunov exponent minimal with respect to φ. We find that $\lambda_{\min}$ is switched to negative in the very large Ω region considered except in the plateau near $\Omega=1$. The plateau can be easily understood since at $\Omega=1$ the second forcing has the same frequency as the first one; then the small second forcing plays only a trivial role in modifying the amplitude and inducing a phase shift in the single-frequency forcing, the latter effect plays no role for the bifurcation figure. Therefore, the dynamics of the system is not significantly changed around $\Omega\approx\omega$.

To end this section, we present some result about the injecting energy of the perturbation. The injecting energy is defined as $E_1=\int_0^T\dot{x}B\cos\omega t\,dt$, $E_2=\int_0^T\dot{x}\alpha B\cos(\Omega t+\varphi)dt$. In Fig. 9(a), we plot the injecting energy of the second forcing versus phase φ at $\Omega=4$, $B=7.15$, and $\alpha=0.1$. From numerical simulation we know that in the range $0.6<\varphi<4.9$ the system is converted to regular motion. It is interesting to see: (i) the absolute value of the input energy of the second forcing is considerably smaller than that of the first forcing [for the energy of the first forcing, see Fig. 9(b)], then we can adjust small energy perturbation to control chaos induced by large energy driving. (ii) The input energy of the second forcing may be positive, negative, or vanishing while that of the first forcing is definitely positive. Therefore, it may happen that one can control chaos without any energy cost, or even with some energy gain.

III. INTERMITTENCY AND BREATHER

In the above section, we discuss the effect of phase on suppressing chaos under exact resonance conditions. However, in experiments we cannot fit the resonance exactly, so there is unavoidable detuning between the two frequencies of the forcings. How a slight detuning influences the system dynamics is an important problem from a practical point of view; that is the main concern of this section.

First, we assume Ω_0/ω to be rational and the frequency of the second forcing $\Omega=\Omega_0+\Delta\Omega$ has a small deviation $\Delta\Omega$ ($|\Delta\Omega|\ll\Omega$) from the resonance. This detuning introduces an actual phase evolution $\varphi(t)=\varphi+\Delta\Omega t$ in Eq. (2). In Fig. 10, we plot the time evolution of the system at $B=8.85$, $\alpha=0.075$, $\Omega_0=3\omega=3$, $\Delta\Omega=1/3000$, and $\varphi=0$; data are obtained on the same surface of section as in Sec. II. In certain time intervals the system moves "regularly" while in other time segments it moves "chaotically," and after a time length $T=2\pi/\Delta\Omega$, the motions are repeated. In "regular" segments, the motion is quasistatically "periodic" and stable against perturbations; however, in "chaotic" segments a very small fluctuation will result in considerable changes of the trajectories. Therefore, in "regular" segments we can predict the system state, while in "chaotic" segments we cannot do so. Though the general dynamic feature of the system repeats after a "period" T, the actual trajectory does not repeat itself after the same time length. This behavior can maintain in a wide range of $\Delta\Omega$ (up to $\Delta\Omega=1/100$ in the case of Fig. 10). Therefore, by introducing a small detuning from resonant frequency we find a new stable "periodic" (in the sense stated above) behavior with period $T=2\pi/\Delta\Omega$ that includes both regular and chaotic motion in its time evolution. We identify this stable "periodic" motion as a breather. The resemblance of Fig. 10 with Fig. 3 is meaningful. A very small

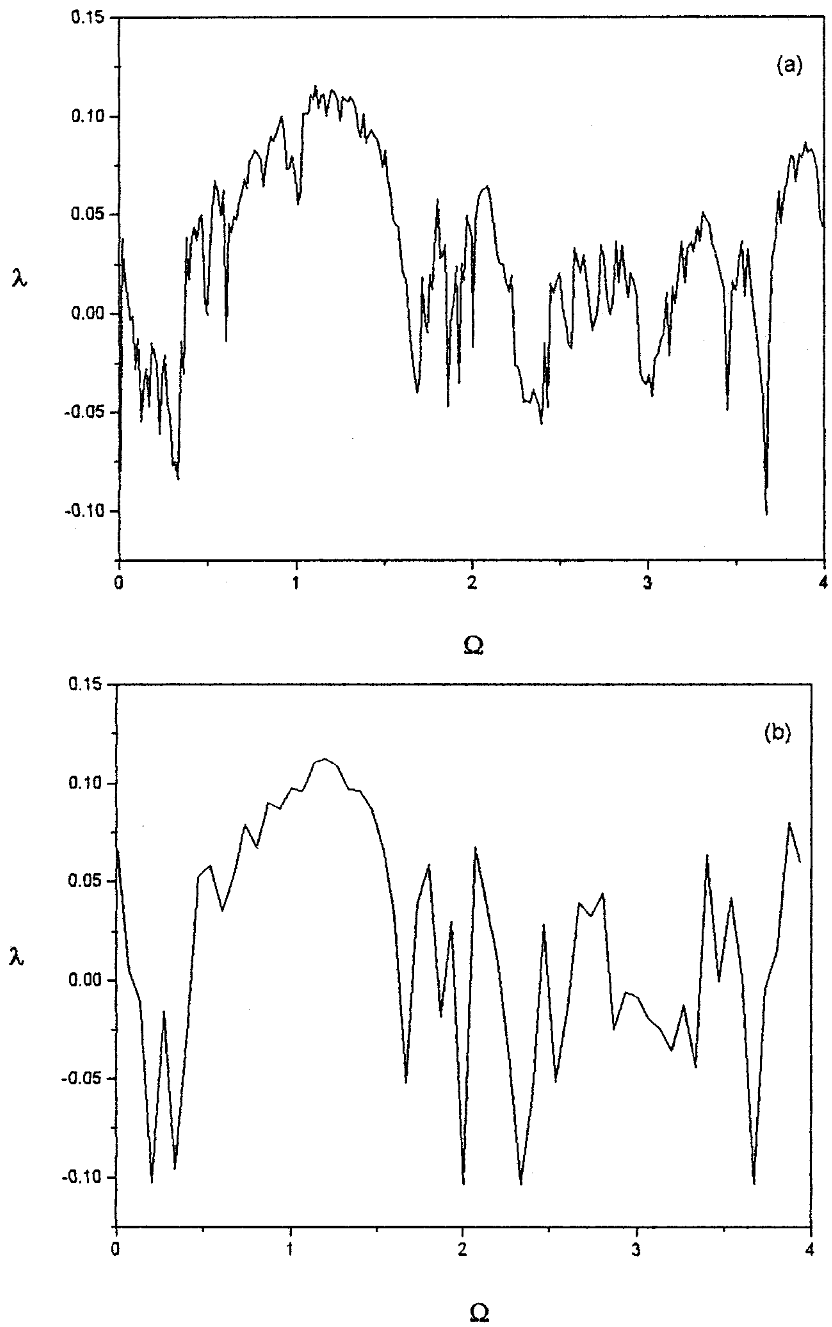

FIG. 8. The leading Lyapunov exponent λ plotted against the frequency of perturbation $B=8.85$, $\alpha=0.1$, (a) $\varphi=0$; (b) $\varphi=2\pi/3$; (c) λ_{min} plotted against Ω. λ_{min} is the minimum λ with respect to φ for other given parameters. λ_{min} is almost entirely suppressed below zero except in the small plateau around $\Omega=1$.

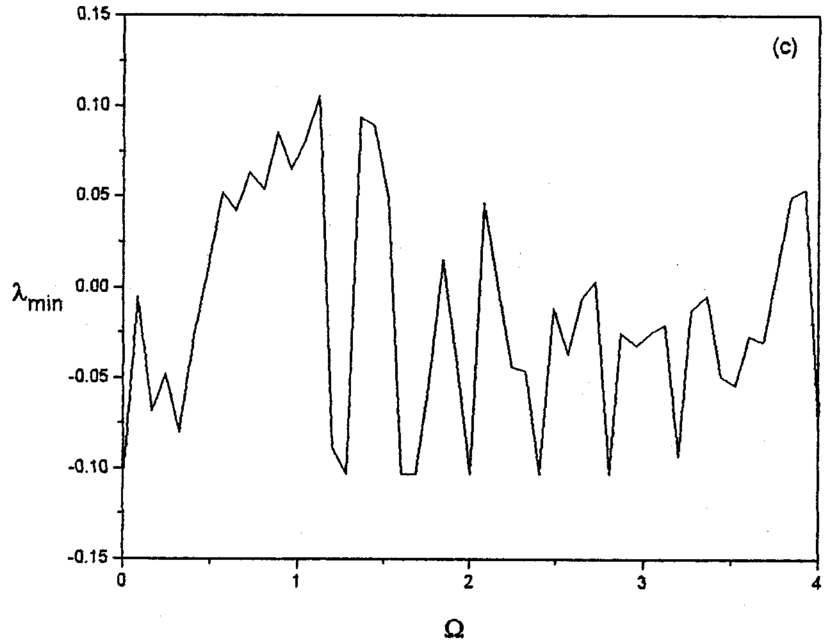

FIG. 8 (*Continued*).

$\Delta\Omega$ introduces a temporary φ, which varies very slowly. As time goes on, the system evolves at different quasistatic φ. The time t in Fig. 10 corresponds to the control parameter φ in Fig. 3, which underlies the resemblance of the two figures.

It is remarkable that we find an interesting state of the system in which chaotic motion and regular motion appear alternatively. This intermittency has an essential difference from conventional intermittencies. In conventional intermittencies, periodic motion and chaotic motion appear irregularly, and their time intervals cannot be predicted. However, in our breatherlike intermittency, the time lengths of chaotic or regular motion are fixed and the time at which chaotic or regular motion appears can be rather precisely predicted. On the other hand, this motion is not quasiperiodic since the adjacent orbits diverge exponentially in the chaotic regions. There are many experiments reporting controlling chaos in nonautonomous systems by applying a small second periodic forcing [12,14]. However, to our knowledge, none of these experiments succeeded in stabilizing periodic orbits steadily. On the contrary, some authors declared that the periodic orbits can be maintained only for a certain time length; avoidable shifts of the bifurcation figure always destroyed the stability of the control. Now, the reason for this instability is clear to us; it is due to the phase effect and the frequency detuning. Based on this understanding, Li *et al.* succeeded in stabilizing periodic orbits and suppressing chaos by using nonfeedback control for an arbitrarily long time, by simply matching the phase of the two forcings [18].

In a practical situation the phase is often subject to noise impacts. Let us assume the phase fluctuates as

$$\dot{\varphi}=\eta(t), \tag{3}$$

where $\eta(t)$ is Gaussian white noise, which satisfies

$$\langle \eta(t)\rangle=0,$$
$$\langle \eta(t)\eta(t')\rangle=2D\delta(t-t'), \tag{4}$$

where D is the intensity of noise. Numerically, the Gaussian white noise $\eta(t)$ is generated by using the Box-Muller method [17]. In Fig. 11, we plot the time evolution of the system with $D=0.007$ and all other parameters given in Fig. 10 except $\Delta\Omega=0$. Now, one can still find breathers of regular or chaotic motions (of course, in the regular parts, some fluctuations naturally induced by noise are observed and high periodicity is wiped by noise as well), but the alternative appearances of both motions are no longer periodic and the changes from one type of motion to another occur irregularly. This behavior resembles conventional intermittency. However, the mechanism underlying this intermittency is the effect of phase and the random walk on the phase; that is different from the mechanisms of conventional intermittencies. If D is too large, the regular part may be hidden, and if D is too small, the system may stay in a certain region (regular or chaotic) for an extremely long time.

IV. MULTIFREQUENCY SYSTEM

In this section we discuss briefly the effect of phase in controlling chaos in multifrequency systems. Specifically, we consider a system driven by two harmonics and subjected to a sinusoidal perturbation. The system is described by Eq. (2) with $B\cos\omega t+\alpha B\cos(\Omega t+\varphi)$ replaced by $B_1\cos\omega_1 t+B_2\cos\omega_2 t+\alpha\cos(\Omega t+\varphi)$, where α is small in comparison with B_i $(i=1,2)$. In Fig. 12 we take $B_1=B_2=15$, $\omega_1=1$, $\omega_2=3$, $\Omega=5$. In Fig. 12(a), we show the bifurcation with respect to amplitude α at a fixed phase $\varphi=0$. In the given range of α, one can see only chaotic motion (of course, period windows must be found by fine resolutions). The small perturbation cannot suppress chaos at the given phase. In Fig. 12(b), we change φ, while fixing α to $\alpha=0.9$, which is much smaller than B_1,B_2 and is also much smaller than the largest α in (a) ($\alpha=4$); a wide regular region emerges in the phase range from $\varphi\approx1$ to 3.5. In this region, the system leaves the chaotic region to periodic motion via inverse pe-

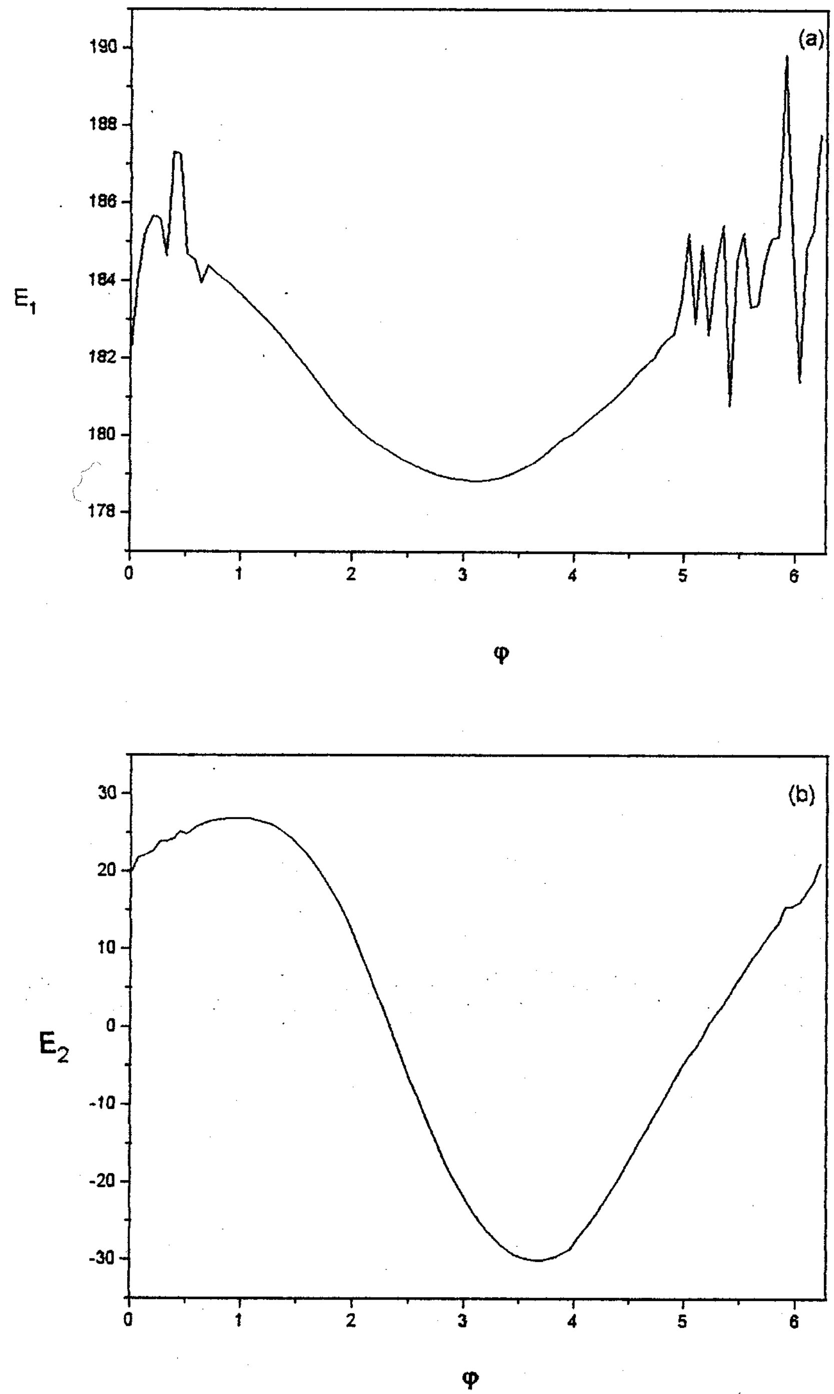

FIG. 9. The injected energy of the first forcing (a) and second forcing (b) plotted against φ. $\Omega=4$, $B=7.15$, and $\alpha=0.1$.

riod doubling. At even smaller α (down to $\alpha=0.4$) we can still control chaos for certain φ. Note in this case that the third perturbation is incomparably smaller than the two driving forces. Thus the phase control of chaos seems to be effective as well in multiharmonic systems.

In fact, an extension of this study to systems with more forcings where more phases can be adjusted will be of much interest. We will go further in this direction in our future works.

In conclusion, we would like to emphasize the following: controlling chaos in systems similar to Eq. (2) was investigated in a number of works [7–9,11,15]. In this paper we thoroughly studied the role played by the phase difference of the two sinusoidal forces and found that the phase term plays a very important role in controlling chaos. Rajasekar [20] studied the effect of the phase difference in controlling chaos by the Melnikov method and numerical simulations, but his discussion focused on the case $\Omega=\omega$ [as we stated previously, applying the second sinusoidal force with $\Omega=\omega$ in Eq. (2) is equivalent to merely changing the amplitude and the initial phase of the first force, and can only shift, not change, the global bifurcation diagram of the system]. Moreover, in

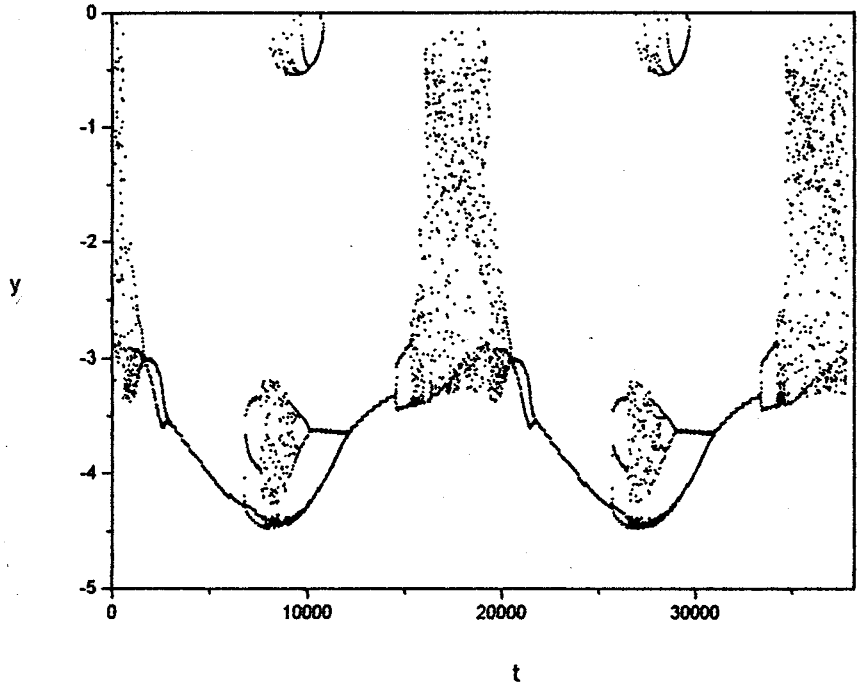

FIG. 10. y plotted against time t for $B=8.85$, $\alpha=0.075$, $\Omega=\Omega_0+\Delta\Omega$, $\Omega_0=3\omega=3$, $\Delta\Omega=1/3000$, and $\varphi=0$. The resemblance of this figure with Fig. 3 is remarkable.

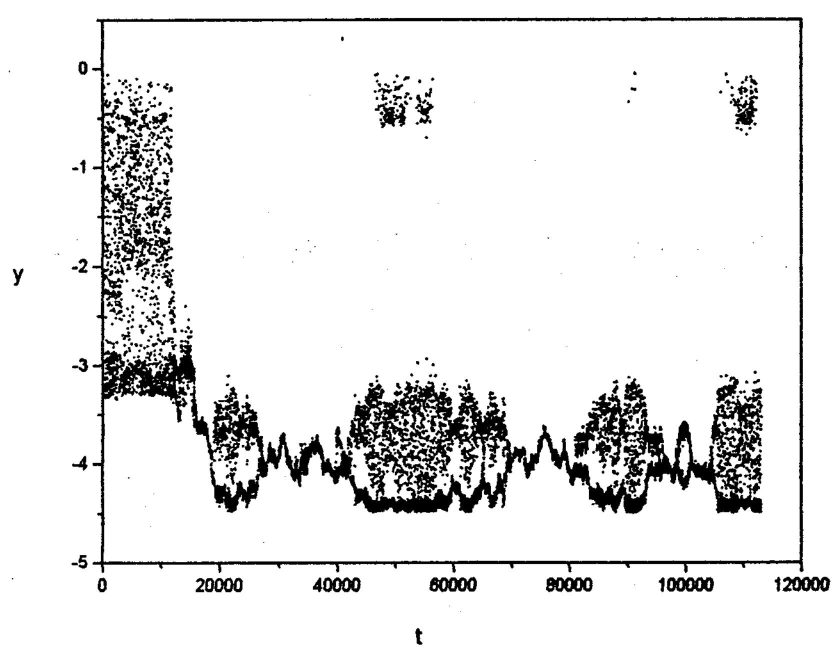

FIG. 11. y plotted against t for $B=8.85$, $\alpha=0.075$, $\Omega=3$, $D=0.007$, and $\varphi=0$.

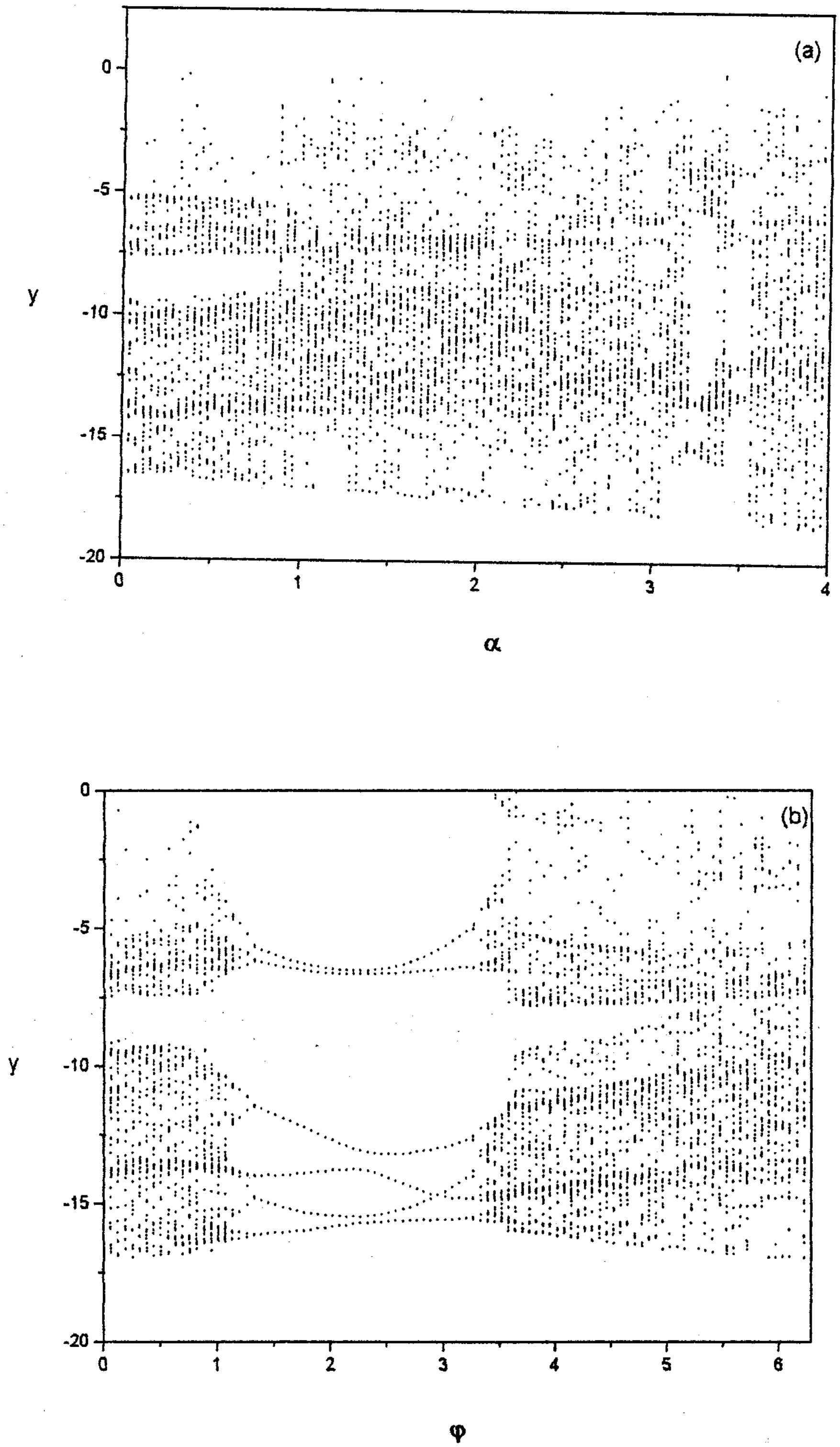

FIG. 12. Triharmonic-forcing Duffing equation simulations. $B_1=15$, $B_2=15$, $\omega_1=1$, $\omega_2=3$. (a) Asymptotic y plotted against α for $\varphi=0$ and $\Omega=5$. Chaos exists in the large region of α. (b) Asymptotic y plotted against φ for $\Omega=5$ and $\alpha=0.9$ [much smaller than the largest α in (a)]. Chaos is suppressed in a large φ interval.

experiments of controlling chaos in systems like Eq. (2), an instability of controlling was reported [12]. Now, based on the understanding of the phase effect, we can explain the mechanism underlying this instability phase shift caused by small frequency detuning. A way to overcome this instability difficulty can be suggested, and proven to be effective.

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